Regge Surfaces and Singularities in a Relativistic Theory

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The analyticity properties of a partial wave amplitude $a(l,s)$ are discussed in a relativistic theory. A domain of holomorphy in l exists when the full amplitude $A(s,t)$ is bounded by a polynomial in t and satisfies the Mandelstam representation. The amplitude $a(l,s)$ is unique provided it is required to satisfy the asymptotic conditions on l needed for Carlson's theorem.

The types of singularity of $a(l,s)$ are studied by tracing them along the Regge surfaces in (l,s) space on which they lie. We divide singularities into different classes that depend on the nature of the Regge surfaces. For one class inelastic (production) processes are unimportant, and we deduce that these singularities must be poles. Another class is related to the possibility of a Regge surface going into an inelastic unphysical sheet via the elastic unphysical sheet. It is shown that knowledge of this remote sheet is very important to the problem of establishing meromorphy domains of $a(l,s)$.

By means of unitarity and dominant Born terms we study the general form of the real sections of Regge surfaces with reference in particular to the occurrence of resonances.

1. INTRODUCTION

In this paper our aim is to clarify some of the problems that arise in studying the partial wave amplitude $a(l,s)$ for complex l and complex s in a relativistic theory. One of the central problems is to discover whether the simple results about meromorphy that were established by Regge[1] for potential scattering also apply in a relativistic theory as suggested by Chew and Frautschi[2] and by Blankenbecler and Goldberger.[3] We study this problem with a view to finding out how it is affected by inelastic (production) processes.

Our starting point is the domain of holomorphy $R(l)\geqslant N$ for $a(l,s)$, which can be established by assuming that the full amplitude $A(s,t)$ satisfies a dispersion relation in t with a finite number N of subtractions. The function $a(l,s)$ is unique provided it is required to be bounded by $\exp(k|l|)$ in $R(l)\geqslant N$, where $0<k<\pi/2$. Then Carlson's theorem[4] can be applied. This holomorphy domain was first established by Gribov[5] and independently by Froissart.[6] The uniqueness problem has been considered by Martin[7] and by Squires.[8] The results are outlined in Sec. 2.

In Sec. 3 we classify the possible singularities of $a(l,s)$ in terms of the properties of the surfaces in complex (l,s) space on which they lie (we refer to them as Regge surfaces). We can study some classes of these singularities by tracking them along suitably chosen paths on their Regge surfaces. These paths are chosen so that they lead into domains of (l,s) space (usually integer l) in which some analyticity properties of $a(l,s)$ have been established. Another class of singularities does not have such paths on its Regge surfaces but may lead instead into the inelastic unphysical sheet corresponding to a production process. We show that it is at least as important to consider the inelastic unphysical sheet reached via the elastic unphysical sheet as it is to consider the one reached directly from the physical sheet. The analyticity properties in these sheets have not yet been established for l complex.

In Sec. 4 we show that information can be obtained about the real sections of Regge surfaces from unitarity and dominant Born exchange terms. The latter are known to dominate near the branch point that they cause and are assumed to dominate at large negative values of the energy. It is shown how two such Born exchange terms (attractive short range and repulsive long range) lead to a Regge curve in the real (l,s) plane that is typical when there is a resonance in the scattering system.

2. ANALYTICITY IN COMPLEX ORBITAL MOMENTUM

For positive integer l, the partial wave amplitude $a_l(s)$ is defined from the amplitude $A(s,z)$ (s is energy squared, z is $\cos\theta$), by the equation,

$$a_l(s)=\frac{1}{2}\int_{-1}^{1} dz\, A(s,z)P_l(z). \tag{2.1}$$

In order to obtain a unique analytic function $a(l,s)$ for complex l when $a_l(s)$ is given at integer values, it is necessary to impose conditions on $a(l,s)$ so that Carlson's theorem[4] can be applied.

Carlson's theorem. If $f(l)$ is regular and of the form $O(\exp k|l|)$, where $k<\pi$, for $R(l)\geqslant N$, and $f(l)=0$ for $l=N,N+1,N+2,\ldots$, then $f(l)=0$ identically.

The required conditions are not satisfied by the integrand in Eq. (2.1); we must, therefore, seek a different integral representation on which to base the ana-

[1] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947, (1960).
[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962).
[3] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).
[4] E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed.
[5] V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 1961 (1962).
[6] M. Froissart (unpublished talk at La Jolla Conference on Theory of Strong Interactions, 1961), see also Phys. Rev. **123**, 1053 (1961).
[7] A. Martin, Phys. Letters **1**, 72 (1962).
[8] E. J. Squires (to be published).

lytic continuation. It will be assumed that, (i) $A(s,z)$ is bounded by a polynomial in z for fixed s including complex values, (ii) $A(s,z)$ satisfies a dispersion relation,

$$A(s,z)=\sum_{p=0}^{N-1} a_p z^p+\frac{z^N}{\pi}\int_{x_0}^{\infty}\frac{dx\,A(s,x)}{x^N(x-z)}+\frac{z^N}{\pi}\int_{-\infty}^{-x_0}\frac{dx\,A(s,x)}{x^N(x-z)}. \quad (2.2)$$

For real negative s the first of these properties has been shown by Froissart[6] to follow from unitarity and the Mandelstam representation. The second also holds if the Mandelstam representation is valid. We consider elsewhere[9] the possibility that for some values of s, with general masses, here may be complex singularities in z that cause a modification to Eq. (2.2).

From Eqs. (2.1) and (2.2), for integer $l>N$,

$$a_l(s)=\frac{1}{\pi}\int_{x_0}^{\infty} dx\,A_t(s,x)Q_l(x)+\frac{1}{\pi}\int_{-\infty}^{-x_0} dx\,A_u(s,x)Q_l(x). \quad (2.3)$$

It has been shown by Gribov,[5] Squires,[8] and Martin[7] that Eq. (2.3) provides a unique method of analytic continuation only for the first integral on the right-hand side. This is because the second integral contains terms like $(-x)^{-l-1}$, which is equal to $x^{-l-1}\exp[-i\pi(l+1)]$. This does not satisfy the boundedness conditions for large l that ensure unique analytic continuation. They note that the difficulty can be overcome by defining a separate analytic continuation from even integer values of l and from odd values. The resulting even and odd partial wave amplitudes of $R(l)\geqslant N$ are given by

$$a^{\pm}(l,s)=\frac{1}{\pi}\int_{x_0}^{\infty} dx\,[A_t(s,x)\pm A_u(s,x)]Q_l(x). \quad (2.4)$$

The even amplitude $a^+(l,s)$ is physically meaningful only for l equal to an even integer or zero, and similarly $a^-(l,s)$ is physically meaningful only for odd integers. The fact that we continue from alternate integers requires that we take $k<\frac{1}{2}\pi$ in Carlson's theorem. This is satisfied by the integrand in Eq. (2.4), and the integral converges in $R(l)\geqslant N$, thus establishing holomorphy in this domain from assumptions (i) and (ii) above. We consider next the question of determining the nature and location of singularities of $a(l,s)$.

3. CONTINUATION ON REGGE SURFACES

In his discussion of potential scattering Regge[1] has considered the paths of the poles of $a(l,s)$ in the l plane as s varies through real values. The importance of these Regge trajectories in a relativistic theory has been stressed by Chew and Frautschi[2] and by Gell-Mann.[10] We show in this section that the types of singularity of $a(l,s)$ can, under certain conditions, be investigated by continuation on the "Regge surfaces" in (l,s) space on which they occur. We use the term singularity for poles or for branch points, but not for branch cuts. Our discussion will be limited to scattering without anomalous thresholds, which for definiteness we illustrate by an equal-mass system. It applies equally to a^+ or a^-, but we do not always use these labels.

The formulas (2.4) for $a^{\pm}(l,s)$ do not define anaytic continuation below $R(l)=\sigma$, where $A(s,x)\sim x^{\sigma}$ for large x. By assumption (i), $N-1<\sigma<N$, and, in general, σ will depend on s. The integral representation (2.4) for $a(l,s)$, happens to be unsuitable for continuation into $R(l)\leqslant\sigma$, but we assume that such a continuation exists and could be obtained if more was known about the nature of the singularities of $a(l,s)$ in this domain. The representation given by Eq. (2.4) is also unsuitable for continuation from $R(s)>4m^2$ to $R(s)<4m^2$, but we assume also in this case that an analytic contination exists since this is unlikely to be a natural boundary.[9]

We divide the possible singularities of $a(l,s)$ into different classes depending on whether they satisfy the following conditions:

Condition A. The location of the singularity of $a(l,s)$ in the complex l plane is a function of s,

$$l=g(s), \quad (3.1)$$

and has an inverse

$$s=f(l). \quad (3.2)$$

We do not require these functions to be single valued.

Condition B. When s is real and above the first inelastic threshold ($9m^2$ for equal masses) the function $g(s)$ is complex, and when s is real and below the elastic threshold ($4m^2$) the function $g(s)$ is real. We require this condition to hold for $g(s)$ when s is on either the physical sheet or on the elastic unphysical sheet.

We now divide the possible singularities of $a(l,s)$ into the following classes:

(I) Singularities that satisfy both conditions A and B.
(II) Singularities that satisfy condition A but not B.
(III) Singularities whose location in the l plane is independent of s.
(IV) Singularities whose location in the s plane is independent of l.
(V) Other singularities.

If the analogy with nonrelativistic potential scattering is valid we would expect some Regge poles to be in class I, possibly even all such poles. In potential scattering a potential that behaves like r^{-2} near the origin produces fixed branch points in the l plane that would be in class III; it is not known whether these are relevant to quantum field theory. In class IV we have singu-

[9] J. L. Challifour and R. J. Eden, Nuovo Cimento (to be published).

[10] M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962).

larities in the s plane due to normal thresholds and the analogous singularities on unphysical sheets (more generally anomalous thresholds would be in this class but we exclude them here). Class II is not relevant to potential scattering. We note also that it is excluded if we assume elastic unitarity since this precludes any consideration of the effects of inelastic processes. In the following discussion we see that class II singularities are related to the possibility of singularities of $a_l(s)$, on the inelastic unphysical sheet for l integer, emerging as singularities of $a(l,s)$ on the physical sheet when l is made complex.

We begin by limiting our discussion to singularities in classes I and IV, and assume that the only relevant class IV singularities are the normal thresholds with their associated branch cuts on the real s axis. We consider a singularity in class I that is at $l_0=g(s_0)$ for real s_0 above the elastic threshold. Trace this singularity on the Regge surface given by Eqs. (3.1) and (3.2) by varying s through real values from s_0 to $s_1=4m^2$, where $l_1=g(s_1)$ becomes real; denote by C_1 the path from l_0 to l_1 in the complex l plane. Next increase l through real values from l_1 to a positive integer (or zero) l_2, causing s to vary on the Regge surface $s=f(l)$ from s_1 to $s_2=f(l_2)$. Call this part of the path C_2.

On the path C_2 the variable l is real, so from condition B either s is real and less than $4m^2$, or s becomes complex. If s becomes complex then C_2 cannot cross the real axis except possibly between $4m^2$ and $9m^2$ by condition B. Therefore, s_2 must be a point that is either in the physical sheet of $a(l_2,s)$ or in the elastic unphysical sheet, provided our starting point (l_0,s_0) referred to a singularity in the physical sheet.

It has been proved for every order in perturbation theory that for equal masses $a(l_2,s)$ ($l_2=$integer), satisfies a dispersion relation in s, so it has no singularities on the physical sheet except the usual branch cuts on the real axis. Hence, the singularity that we have traced to (l_2,s_2) must refer to a point in the elastic unphysical sheet. On this sheet the amplitude is given by (Blankenbecler *et al.*[11]),

$$a^{(2)}(l_2,s)=a^{(1)}(l_2,s)\{S^{(1)}(l_2,s)\}^{-1}, \tag{3.3}$$

where the superscripts (1) and (2) denote the functions on the physical and elastic unphysical sheets, respectively. The only singularities of $a^{(2)}(l_2,s)$ come from the zeros of the S-matrix element and are, therefore, poles. It follows that the singularity of class I that we have traced to (l_2,s_2) must be a pole. It was, therefore, a pole at l_0 in the complex l plane for s equal to the initial value s_0, since the type of singularity does not change on its Regge surface. A similar argument holds if as l increases from l_1 to l_2 the point s moves along the real axis to $s_2<4m^2$ where it must be associated with a bound-state pole or a virtual-state pole.

[11] R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961).

There are two points of difficulty in the above argument that require some care. The first relates to the possibility that the singularity of $a(l,s)$ happens to vanish when we reach (l_2,s_2) (as for example with a pole of zero residue or a branch cut of zero discontinuity). Then we could not conclude from our knowledge about $a(l_2,s)$, whether the original singularity was a pole or a branch cut. However, this difficulty of a vanishing singularity must persist at every integer if it is to prevent our method from working. Then we can apply Carlson's theorem to show that the associated function that vanishes at each integer must vanish everywhere. Hence, the boundedness assumption on the amplitude prevents this difficulty from arising.[12] The second point of difficulty concerns the possibility of multiple singularities of class I. Then as we follow the path C_1+C_2 the singularity might go through a branch cut in the l plane of $a(l,s)$ due to another class I singularity. Such a branch cut can be distorted away from C_1+C_2 unless we go through the branch point itself. If this happens we then transfer our attention to the new branch point and follow this along a modified path $C_1'+C_2'$ which is partly on one Regge surface and partly in another. We again reach the point (l_2,s_2) in the elastic unphysical sheet and deduce that the singularity cannot be a branch point but must be a pole.

In the above discussion we have assumed that the path followed on the Regge surface does not go to infinity. This can be ensured since $f(l)$ and $g(s)$ are analytic functions and a small detour on the Regge surface will always suffice to avoid the point at infinity in either complex plane. If $f(l)$ and $g(s)$ are not analtyic, the singularity would be in class V and is not considered here.

We discuss now the problem of continuation of singularities of class II along their Regge surfaces. We show, in particular, that they are related to the possible singularities of $a(l_2,s)$, $l_2=$integer, in one of the inelastic unphysical sheets. The latter may be reached either directly from the physical sheet or from the elastic unphysical sheet. The latter possibility makes the use of "physical arguments" about the finiteness of the amplitude for real s and l somewhat difficult. In the simple problem of real s in the elastic region ($4m^2<s<9m^2$), we can use generalized unitarity,

$$S(l,s)S^*(l^*,s)=1. \tag{3.4}$$

This shows that a Regge surface $l=g(s)$ cannot have l and s simultaneously real in the elastic region unless S becomes undefined (zero over zero). The latter possibility does not affect our conclusions so we do not discuss it here. Hence the curve C_1+C_2 that we discussed above will cross the real s axis only at the elastic threshold. This is illustrated in Fig. 1(a).

Now consider the corresponding curve C_1+C_2 for

[12] We are indebted to P. V. Landshoff for raising this difficulty and to E. J. Squires for settling it.

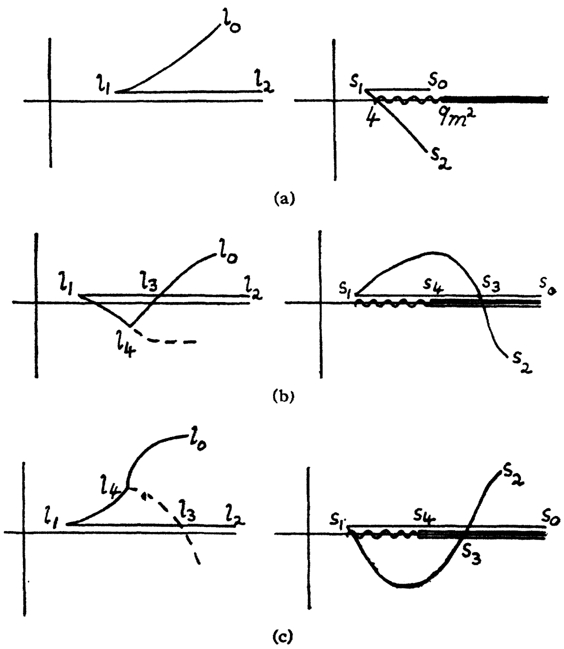

FIG. 1. The paths traced along Regge surfaces for class I singularities (a), and for class II singularities (b) and (c).

class II singularities, beginning at (s_0,l_0) with $s_0>9m^2$, moving to $(s_1=4m^2,l_1)$ and then to (s_2,l_2) with l_2 an integer. The two main possibilities are illustrated in Figs. 1(b) and (c). The Regge surface will have a branch point at the inelastic threshold $s=9m^2$, this is denoted by the points l_4 and s_4 in Figs. 1(b) and (c). The point where the curve C_2 crosses the real s axis (l real) is denoted (l_3,s_3).

The path C_1+C_2 in Fig. 1(b) indicates that the real point (l_3,s_3) is a singularity of $a(l,s)$ for l real as $s+i\epsilon\rightarrow s$ from the physical sheet. If l_3 was an integer this would correspond to a singularity of the physical partial wave amplitude and could be excluded, but for a general real value of l_3 this cannot be done.

For the curve C_1+C_2 shown in Fig. 1(c) the real point (l_3,s_3) is a singularity of $a(l,s)$ for l real in the limit $s-i\epsilon\rightarrow s$ from the elastic unphysical sheet. However, if we extend Eq. (3.3) from integer l_2 to complex values of l, we see that a branch point on the elastic unphysical sheet will in general imply that there is a branch point also on the physical sheet. This suggests that it may be sufficient to investigate the inelastic unphysical sheet reached directly from the physical sheet.

In principle the analytic properties of the amplitude $a(l,s)$ can be studied on the inelastic unphysical sheets by generalizing Eq. (3.3). This generalization has not yet been done in sufficient detail to indicate whether any singularities there, are in class II. Some calculations by Gribov (private communication) suggest that continuation from the physical sheet to the inelastic unphysical sheet, Fig. 1(b), leads to complex singularities in class IV. Continuation along the path in Fig. 1(c) has not yet been studied. Pending an investigation of these unphysical sheets it is not possible to deduce the validity of the hypothesis that $a(l,s)$ is meromorphic in the l plane in relativistic theory. We note also that additional complications would arise from branch cuts fixed in the l plane since these might interfere with our tracking procedure on Regge surfaces by introducing additional unphysical sheets.

4. REAL SECTION OF A REGGE SURFACE

In this section we study some consequences of Eq. (3.3) which relates the singularities of $a^{(2)}(l,s)$ on the elastic unphysical sheet to the zeros of $S^{(1)}(l,s)$ on the physical sheet. We establish that the odd and even parts of $a(l,s)$ do not have the same singularities, and discuss circumstances that could lead to resonances. In the first instance, our discussion is limited to the scattering of equal mass particles (mass M) that may exchange particles of mass m. Later we will consider also the exchange of particles of different masses. The inclusion of Born exchange terms leads to an obvious modification of Eq. (2.4) defining $a^{\pm}(l,s)$.

Our work is based on the following results that were obtained from Eq. (3.3) for integer values of l by Blankenbecler *et al.*[11] The partial wave amplitude $a_l(s)$ is analytic in the physical sheet except for branch cuts,

$$s<4M^2-m^2,\quad\text{and}\quad 4M^2<s. \tag{4.1}$$

The branch point at $4M^2-m^2$ comes from the Born term in the t channel, for exchange of a single particle of mass m. The S-matrix element $S_l(s)$ is unity at the normal threshold $s=4M^2$, and near $(4M^2-m^2)$ it is dominated by the Born exchange term. Thus,

$$S_l(s)\rightarrow g(-1)^{l+1}\text{ times plus infinity, as}\quad s\rightarrow(4M^2-m^2), \tag{4.2}$$

where g is positive if the Born term corresponds to an attractive potential, and negative for a repulsive one. If there is no bound state, $S_l(s)$ is bounded in the gap between the branch cuts. Therefore, with g positive, $S_1(s)$ has an odd or even number of zeros in the gap depending on whether l is even or odd, respectively.

If there is one bound state then $a_l(s)$ has a pole with positive residue on the physical sheet and $S_l(s)$ has the same pole. Then for positive $g,S_l(s)$ has an odd or even number of zeros depending on whether l is odd or even. This illustrates how a bound state corresponds to a pole that migrates to the unphysical sheet if the attractive force is weakened.

The even and odd parts $a^+(l,s)$ and $a^-(l,s)$ correspond to $a_l(s)$ at even and odd integers, respectively. The poles of $a^{\pm}(l,s)$ lie on Regge surfaces whoes real sections are curves in the real l, real s plane. If we assume that they are class I singularities in the sense described in the previous section, the curves will go complex at $s=4M^2$. If we assume also that the Born exchange term dominates at large negative values of s, the curves will tend asymptotically to negative integer values of l. We deduce for an attractive Born term that for integer l:

(1) $a^{+(1)}(l,s)$ and $a^{+(2)}(l,s)$ taken together have an odd number of poles in the gap $(4M^2-m^2<s<4M^2)$.

(2) $a^{-(1)}(l,s)$ and $a^{-(2)}(l,s)$ taken together have an even number of zeros in the gap.

Typical Regge curves (the real sections of Regge surfaces) for an attractive potential are illustrated in Fig. 2—(a) for the even amplitude, and (b) for the odd amplitude. The curves shown as broken lines are the real projections from (l real, s complex) of the Regge surface on the second sheet (note s becomes real when l is an integer on this surface). Since these curves must have different number of intersections in the gap it is evident that the even and odd amplitudes do not have the same singularities. This implies in Eq. (2.4) that the contributions from the left- and right-hand cuts (from A_t and A_u) will normally both be singular at every Regge pole.

The curves in Figs. 2(a), (b), correspond to a single attractive Born term and are, therefore, not of resonant type. For a resonance it is necessary for a broken curve to double back on itself as illustrated in Fig. 2(c). The corresponding Regge trajectory is complex where the real section has a maximum in l. The projections of the complex parts of the Regge trajectory, and of the Breit-Wigner trajectory are shown as broken lines in Fig. 2(c). We consider next the way that two Born terms corresponding to exchanged particles of different masses can lead to this type of Regge curve.

A typical resonance will come from a long-range repulsion and a shorter range attraction. In Fig. 2(c) we have drawn a broken curve for $a^+(l,s)$ that corresponds to a repulsion for the exchange of a particle of mass m_1, and an attraction with exchange of m_2, there $m_1<m_2$. From our previous discussion there must be an even number of intersections with $l=$integer in the gap $(4M^2-m_1^2<s<M^2)$. Also since $S_1^+(s)$ changes sign between $(4M^2-m_1^2)$ and $(4M^2-m_2^2)$, there must be an odd number of intersections in this interval. This forces the broken curve to have the form shown in Fig. 2(c). A second curve is also drawn below the first, this would be expected to have a less pronounced resonance since it corresponds to a state with more oscillations in the radial wave function. In each case the curves will change from the physical sheet to the elastic unphysical sheet at the tangent point to $s=4M^2$.

The foregoing representation of Regge curves and their real projections, can readily be adapted to include the exchange of pairs of particles in resonant states, though some numerical work would be encountered in determining whether the resonance pole would dominate so as to ensure a zero of the S-matrix element. It will also be possible to estimate the sharpness of any associated resonance from the form of the real Regge curve near threshold. It is hoped that these numerical questions will be studied in another paper.

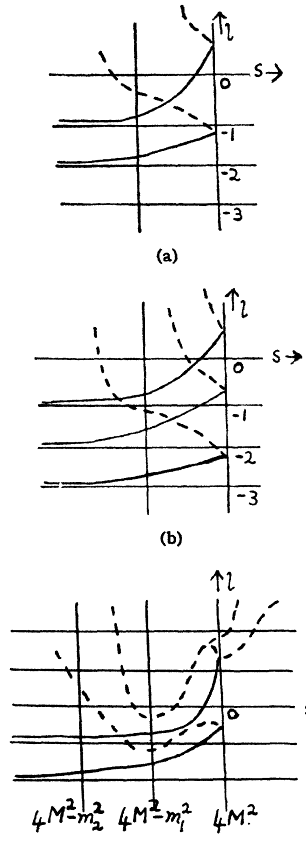

FIG. 2. Regge curves in the real (l,s) plane; (a) and (b) for even and odd amplitudes, respectively, and (c) for an even amplitude in a resonant case.

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